

Discontinuities of multi-Regge amplitudes

V.S. Fadin

Budker Institute of Nuclear Physics of SD RAS and Novosibirsk State University, 630090 Novosibirsk, Russia

Abstract. In the BFKL approach, discontinuities of multiple production amplitudes in invariant masses of produced particles are discussed. It turns out that they are in evident contradiction with the BDS ansatz for n -gluon amplitudes in the planar $N=4$ SYM at $n \geq 6$. An explicit expression for the NLO discontinuity of the two-to-four amplitude in the invariant mass of two produced gluons is presented.

Keywords: BFKL equation, multi-Regge kinematics, discontinuities of MRK amplitudes, BDS ansatz

PACS: 11.15.-q, 12.38.Cy

INTRODUCTION

The BFKL (Balitsky-Fadin-Kuraev-Lipatov) equation [1] was derived using the assumption of the multi-Regge form of multiple production amplitudes. According to this assumption, the amplitude $\mathcal{A}_{2 \rightarrow 2+n}$ of the production of the gluons G_1, G_2, \dots, G_n in the collision of particles (gluons or quarks) A and B in the multi-Regge kinematics (MRK), which means strong ordering of longitudinal momenta and limitation of transverse momenta, takes the form

$$\mathcal{A}_{2 \rightarrow 2+n} = 2s \Gamma_{A'A}^{R_1} \frac{(s_1/s_1^0)^{\omega(t_1)}}{t_1} \gamma_{R_1 R_2}^{G_1} \frac{(s_2/s_2^0)^{\omega(t_2)}}{t_2} \gamma_{R_2 R_3}^{G_2} \dots \frac{(s_n/s_{n+1}^0)^{\omega(t_n)}}{t_n} \gamma_{R_n R_{n+1}}^{G_n} \frac{(s_{n+1}/s_0)^{\omega(t_{n+1})}}{t_{n+1}} \Gamma_{B'B}^{R_{n+1}}, \quad (1)$$

where $\omega(t) = j(t)$, $j(t)$ is the gluon Regge trajectory, $\Gamma_{P'P}^R$ and $\gamma_{RR'}^P$ are scattering and production Reggeon vertices,

$$s = (p_A + p_B)^2, \quad s_i = (k_{i-1} + k_i)^2, \quad t_i = q_i^2, \quad i = 1, \dots, n+1, \quad k_0 = p_{A'}, \quad k_{n+1} = p_{B'}, \quad k_l = p_{G_l}, \quad l = 1, \dots, n,$$

$$q_1 = p_A - p_{A'}, \quad q_{l+1} = q_l - k_l, \quad s \gg s_i \gg |t_i|, \quad (2)$$

and s_i^0 are energy scales. They are not important in the leading logarithmic approximation (LLA), but must be agreed with the Reggeon vertices in the next-to-leading approximation (NLA).

Note that in the NLA the simple factorized multi-Regge form (1) is valid only for real parts of the MRK amplitudes. Fortunately, only these parts are necessary for derivation of the BFKL equation in the NLA. Note also that for the derivation in the NLA, the multi-Regge form is supposed also for production of a couple of particles with limited invariant mass instead of one of the particles A', G_1, G_2, \dots, B' .

The assumption of the multi-Regge form is extremely strong since an infinite number of production amplitudes is expressed in terms of the gluon Regge trajectory and several Reggeon vertices. It was proved lately in the LLA [2] and in the NLA (see [3] and references therein).

DISCONTINUITIES OF MULTI-REGGE AMPLITUDES

The real parts of the amplitudes are expressed through the s_{ij} -channel discontinuities of these amplitudes ($s_{ij} = (k_i + k_j)^2$), so that it is possible to say that the discontinuities are more important than the real parts. Remind that the equation for the BFKL Pomeron is derived from consideration of the s -channel discontinuities of the elastic amplitudes. Note also that the discontinuities are more complex than the real parts. Even the simplest of them, the discontinuities of the elastic amplitudes, have not a simple factorized form. Instead, they are given by the convolution of the particle-particle impact factors and the Green's functions of two interacting Reggeized gluons, which are determined by the BFKL equation.

Evidently, the s_{ij} -channel discontinuities of multi-particle production amplitudes are even more complex. In comparison with the s -channel discontinuities of the elastic amplitudes, they contain two additional components [4]: impact factors for Reggeon-gluon transitions and matrix elements of the gluon production operator

between two-Reggeon states. These components are expressed through effective vertices describing interaction of Reggeized gluons with ordinary gluons and quarks. Now all of them are known in the next-to-leading order (NLO).

Proof of the multi-Regge form

The most known application of the s_{ij} -channel discontinuities is the proof of the multi-Regge form of QCD amplitudes [4]. Compatibility of the s -channel unitarity with the multi-Regge form leads to the bootstrap relations connecting discontinuities of the amplitudes with their real parts and the gluon trajectories:

$$\frac{-1}{2\pi i} \left(\sum_{l=j+1}^{n+1} \Delta_{jl} - \sum_{l=0}^{j-1} \Delta_{lj} \right) = \frac{1}{2} (\omega(t_{j+1}) - \omega(t_j)) \Re \mathcal{A}_{2 \rightarrow 2+n}. \quad (3)$$

Here Δ_{ij} are the discontinuities of $\mathcal{A}_{2 \rightarrow 2+n}$ in the s_{ij} channels, which must be calculated using the Reggeized form of amplitudes in the unitarity conditions.

It turns out that fulfilment of an infinite set of the bootstrap relations guarantees the multi-Regge form of scattering amplitudes. On the other hand, all bootstrap relations are fulfilled if several conditions imposed on the Reggeon vertices and the trajectory (bootstrap conditions) hold true. Fulfilment of all these conditions is proved now (see [3] and references therein) not only in Quantum Chromodynamics (QCD), but in Yang-Mills theories containing fermions and scalars in arbitrary representations of the colour group with any Yukawa-type interaction.

There are other applications of the discontinuities, related to the BDS (Bern, Dixon, Smirnov) ansatz [5] for amplitudes with maximal helicity violation (MHV) in $N = 4$ supersymmetric Yang-Mills theory with large number of colours N_c (in the planar approximation). One of them is a simple proof of violation of this ansatz for n -gluon amplitudes at $n \geq 6$.

Violation of the BDS ansatz

Indeed, let us consider the s_2 - channel discontinuity of the amplitude $A_{2 \rightarrow 4}$ of the process

$$A + B \rightarrow A' + G_1 + G_2 + B', \quad (p_{A'} + p_{G_1})^2 = s_1, \quad (p_{G_1} + p_{G_2})^2 = s_2, \quad (p_{G_2} + p_{B'})^2 = s_3, \quad (4)$$

$$p_A - p_{A'} = q_1, \quad p_{B'} - p_B = q_3, \quad p_A - p_{A'} - p_{G_1} = p_{B'} + p_{G_2} - p_B = q_2$$

in the multi-Regge kinematics $s \gg s_i \gg -q_i^2 \simeq \vec{q}_i^2$ (\vec{q}_i denotes transverse to the (p_A, p_B) plane components of q_i). In the BDS ansatz dependence of this discontinuity on the energy variables s_i is determined by the product of the Regge factors $(s_1)^{\omega(t_1)} (s_2)^{\omega(t_2)} (s_3)^{\omega(t_3)}$ ($t_1 \equiv q_1^2$). But in the BFKL approach, according to [4], it contains additional factor

$$\langle G_1 R_1 | e^{\hat{K}_m \ln \left(\frac{s_2}{|k_1||k_2|} \right)} | G_2 R_3 \rangle, \quad (5)$$

where $\langle G_1 R_1 |$ and $| G_2 R_3 \rangle$ are the impact factors for Reggeon-gluon transitions, $\hat{K}_m \equiv \hat{K} - \omega(t_2)$, \hat{K} is the BFKL kernel in the adjoint representation of the colour group. Therefore, for agreement with the BDS, the impact factors for Reggeon-gluon transitions must be proportional to the eigenfunction of the BFKL kernel with the eigenvalue equal to the gluon trajectory, that evidently contradicts the bootstrap conditions. Indeed, it follows from these conditions that such eigenfunction is proportional to the impact factors for particle-particle transitions, not for Reggeon-gluon transitions. The last ones evidently differ from the first already in the leading order.

It is worthwhile to note that the contradiction described above appears only for n -gluon amplitudes with $n \geq 6$, because the BFKL discontinuities of amplitudes with $n < 6$ contain, instead of (5), only matrix elements where at least one of the impact factors describes particle-particle transition.

In fact, the incompleteness of the BDS ansatz at $n \geq 6$ is well known. The first indications of the incompleteness were obtained in [6] in the strong coupling regime using the Maldacena hypothesis [7] about ADS/CFT duality and in [8] using the hypothesis of scattering amplitude/Wilson loop correspondence. Then the incompleteness was shown by direct two-loop calculations in [9].

Moreover, disagreement of the BDS ansatz with the BFKL approach is also known [10]. Dignity of the consideration presented here is its simplicity, incomparable with the rather sophisticated analysis performed in [10].

The s_2 -channel discontinuity of the $\mathcal{A}_{2 \rightarrow 4}$ amplitude in the NLO

Another application of the discontinuities is to test the hypothesis of dual conformal invariance [11], which states that the MHV amplitudes are given by the products of the BDS amplitudes and the remainder functions depending only on the anharmonic ratios of kinematic invariants, and the hypothesis of scattering amplitude/Wilson loop correspondence [12], which states that the remainder functions are given by expectation values of Wilson loops. Both these hypothesis are not proved. They can be tested by comparison of the BFKL discontinuities with the discontinuities calculated with their use [13, 14].

The BFKL discontinuities contain the matrix element (5). It can be calculated in the NLA using the eigenvalues of the kernel \hat{K}_m obtained in [14], existence of the representation where the kernel is invariant with respect to the Möbius transformations in the transverse momentum space proved in [15] and the Reggeon-gluon impact factors in this representation found in [16]. Using these results, we obtain [17] in the NLA for production of gluons with positive helicities

$$\begin{aligned} \langle G_1 R_1 | e^{\hat{K}_m \ln \left(\frac{s_2}{|\vec{k}_1| |\vec{k}_2|} \right)} | G_2 R_3 \rangle &= \delta(\vec{q}_1 - \vec{k}_1 - \vec{k}_2 - \vec{q}_3) g^4 N_c \frac{q_1^- q_3^+}{k_1^+ k_2^-} \\ &\times \left[1 + \frac{g^2 N_c \Gamma(1-\varepsilon)}{(4\pi)^{2+\varepsilon}} \left(-\frac{1}{2} \ln^2 \left(\frac{\vec{q}_1^2}{\vec{q}_2^2} \right) - \frac{(\vec{k}_1^2)^\varepsilon}{\varepsilon^2} - \frac{1}{2} \ln^2 \left(\frac{\vec{q}_3^2}{\vec{q}_2^2} \right) - \frac{(\vec{k}_2^2)^\varepsilon}{\varepsilon^2} + 4\zeta(2) \right) \right] \\ &\times \left[\frac{1}{2} \sum_{n=-\infty}^{+\infty} (-1)^n \int_{-\infty}^{+\infty} d\nu \left(e^{\omega(\nu, n) \ln \left(\frac{s_2 \vec{q}_2^2}{|\vec{q}_1| |\vec{q}_3| |\vec{k}_1| |\vec{k}_2|} \right)} - 1 \right) w^{\frac{n}{2} + i\nu} (w^*)^{-\frac{n}{2} + i\nu} \right. \\ &\int \frac{dz_1}{\pi |z_1|^2} \frac{1}{1-z_1} \left(1 + \frac{g^2 N_c}{16\pi^2} I(z_1) \right) z_1^{\frac{n}{2} + i\nu} (z_1^*)^{-\frac{n}{2} + i\nu} \int \frac{dz_2}{\pi |z_2|^2} \frac{1}{1-z_2^*} \left(1 + \frac{g^2 N_c}{16\pi^2} I^*(z_2) \right) (z_2^*)^{\frac{n}{2} - i\nu} z_2^{-\frac{n}{2} - i\nu} \\ &\left. + \frac{k_1^+ k_2^-}{q_3^+ q_1^-} \int \frac{d\vec{r}}{\vec{r}^2 (\vec{q}_2 - \vec{r})^2} \left(q_1 - (q_1 - r) \frac{\vec{q}_1^2}{(\vec{q}_1 - \vec{r})^2} \right)^- \left(-q_3 + (q_3 - r) \frac{\vec{q}_3^2}{(\vec{q}_3 - \vec{r})^2} \right)^+ \right], \end{aligned} \quad (6)$$

where $a^\pm = a_x \pm ia_y$ for any two-dimensional vector \vec{a} , $\varepsilon = (D-4)/2$, D is the space-time dimension, $w = k_1^+ q_3^+ / (k_2^+ q_1^+)$,

$$\begin{aligned} I(z) &= \frac{1-z}{8} \left(\ln \left(\frac{|1-z|^2}{|z|^2} \right) \ln \left(\frac{|1-z|^4}{|z|^6} \right) - 6\text{Li}_2(z) + 6\text{Li}_2(z^*) \right. \\ &\quad \left. - 3 \ln |z|^2 \ln \frac{1-z}{1-z^*} \right) - \frac{1}{2} \ln |1-z|^2 \ln \frac{|1-z|^2}{|z|^2} - \frac{3}{8} \ln^2 |z|^2, \end{aligned} \quad (7)$$

$$\begin{aligned} \omega(\nu, n) &= \frac{g^2 N_c}{8\pi^2} \left(\frac{1}{2} \frac{|n|}{\nu^2 + \frac{n^2}{4}} - \psi(1+i\nu - \frac{|n|}{2}) + \psi(1-i\nu + \frac{|n|}{2}) + 2\psi(1) \right) \left(1 - \frac{g^2 N_c}{8\pi^2} \zeta(2) \right) \\ &+ \left(\frac{g^2 N_c}{8\pi^2} \right)^2 \left(\frac{1}{4} \left(\psi''(1+i\nu + \frac{|n|}{2}) + \psi''(1-i\nu + \frac{|n|}{2}) \right) \frac{2i\nu \left(\psi'(1-i\nu + \frac{|n|}{2}) - \psi'(1+i\nu + \frac{|n|}{2}) \right)}{\nu^2 + \frac{n^2}{4}} \right. \\ &\quad \left. + 3\zeta(3) + \frac{1}{4} \frac{|n| \left(\nu^2 - \frac{n^2}{4} \right)}{\left(\nu^2 + \frac{n^2}{4} \right)^3} \right). \end{aligned} \quad (8)$$

Here $\psi(x) = (\ln \Gamma(x))'$, $\zeta(n)$ is Riemann zeta function,

$$Li_2(z) = - \int_0^1 \frac{dx}{x} (1-xz). \quad (9)$$

Note, that $\omega(\nu, n)$ has the important property

$$\omega(0, 0) = 0 \quad (10)$$

in accordance with the bootstrap conditions.

Compatibility of this result with the BDS ansatz corrected by the remainder factor calculated using the dual conformal invariance and scattering amplitude/Wilson loop correspondence hypotheses is under consideration.

SUMMARY

The s_{ij} -channel discontinuities of the MRK amplitudes are more complex objects than the real parts of these amplitudes. They are expressed in terms of the BFKL kernel, the impact factors for particle-particle and Reggeon-gluon transitions and the matrix elements of the gluon production operator between two-Reggeon states. All ingredients entering in this expressions are known now in a closed form.

The most known application of the discontinuities is the proof of the multi-Regge form of the MRK amplitudes. Now this form is proved in theories with fermions and scalars in arbitrary representations of the colour group with any Yukawa-type interaction.

Knowledge of the discontinuities permits to check the BDS ansatz and the hypotheses about the remainder functions to this ansatz.

The BFKL discontinuities are in an evident contradiction with the BDS ansatz for $2 \rightarrow 2 + n$ amplitudes at $n \geq 2$.

To check the hypotheses about the remainder functions we calculated the matrix element (5) in the NLA. Comparison of the result (6) with the results obtained using these hypotheses is under consideration.

ACKNOWLEDGMENTS

The work is supported by the Ministry of Education and Science of the Russian Federation and by the RFBR grant 13-02-01023. The participation of V.F. to Diffraction 2014 was partially supported by HadronPhysics3 within the Seventh Framework Programme (FP7) of EU.

REFERENCES

1. V. S. Fadin, E. A. Kuraev and L. N. Lipatov, Phys. Lett. B **60** (1975) 50; E. A. Kuraev, L. N. Lipatov and V. S. Fadin, Sov. Phys. JETP **44** (1976) 443, *ibid.* **45** (1977) 199; I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822.
2. Ya. Ya. Balitskii, L.N. Lipatov and V.S. Fadin, in *Materials of IV Winter School of LNPI* (Leningrad, 1979) 109.
3. M.G. Kozlov, A.V. Reznichenko and V.S. Fadin, Phys. Atom. Nucl. **77** (2014) 251.
4. V. S. Fadin, R. Fiore, M. G. Kozlov and A. V. Reznichenko, Phys. Lett. B **639** (2006) 74.
5. Z. Bern, L.J. Dixon and V.A. Smirnov, Phys. Rev. D **72** (2005) 085001 [hep-th/0505205].
6. L. F. Alday and J. Maldacena, JHEP **0711** (2007) 068 [arXiv:0710.1060 [hep-th]].
7. J. M. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200].
8. J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Phys. Lett. B **662** (2008) 456 [arXiv:0712.4138 [hep-th]].
9. Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. D **78** (2008) 045007 [arXiv:0803.1465 [hep-th]].
10. J. Bartels, L. N. Lipatov and A. Sabio Vera, Phys. Rev. D **80** (2009) 045002 [arXiv:0802.2065 [hep-th]].
11. J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B **826** (2010) 337 [arXiv:0712.1223 [hep-th]].
12. J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B **795** (2008) 52 [arXiv:0709.2368 [hep-th]].
13. L.N. Lipatov and A. Prygarin, Phys. Rev. D **83** (2011) 045020; *ibid.* **83** (2011) 125001; J. Bartels, L.N. Lipatov and A. Prygarin, Phys. Lett. B **705** (2011) 507.
14. V.S. Fadin and L.N. Lipatov, Phys. Lett. B **706** (2012) 470 [arXiv:1111.0782 [hep-th]].
15. V. S. Fadin, R. Fiore, L. N. Lipatov and A. Papa, Nucl. Phys. B **874** (2013) 230.
16. V. S. Fadin and R. Fiore, arXiv:1402.5260 [hep-th].
17. V. S. Fadin and R. Fiore, to be published.